Asymptotically Optimal Thickness Bounds of Generalized Bar Visibility Graphs

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Abstract

Given a set of disjoint horizontal line segments (call bars), the distance of two bars is the minimum number of the other bars that a vertical line segment joining the two bars passes through. A graph \( G \) is a \( \text{bar } k \)-visibility graph if \( G \) can be represented as a set of disjoint bars such that two vertices are adjacent in \( G \) if and only if the distance of their associated bars is at most \( k \). A graph \( G \) is a \( \text{semi bar } k \)-visibility graph if \( G \) can be represented as a set of disjoint bars whose left endpoints have the same \( x \)-coordinates such that two vertices are adjacent in \( G \) if and only if the distance of their associated bars is at most \( k \). The thickness of \( G \) is the minimum number of planar subgraphs whose union is \( G \).

Dean et al. gave the best previously known upper bound \( 3k(6k+1) \) on the thickness of bar \( k \)-visibility graphs. Hartke et al. proved that \( K_{4k+4} \) is a bar \( k \)-visibility graph, so the upper bound on the thickness of bar \( k \)-visibility graphs is at least \( \lceil(2k+3)/3 \rceil \). Felsner and Massow gave an upper bound on the thickness of semi bar \( 1 \)-visibility graphs. Felsner and Massow proved that \( K_{2k+3} \) is a semi bar \( k \) visibility graph, so the upper bound on the thickness of semi bar \( k \) visibility graphs is at least \( \lceil(2k+5)/6 \rceil \). We reduce the upper bound to \( 3k+3 \) on the thickness of bar \( k \)-visibility graphs, and give an upper bound \( 2k \) for semi bar \( k \)-visibility graphs.

1 Introduction

All graphs are simple throughout the paper. Consider a set \( B \) of disjoint bars, that is, horizontal line segments. For any two bars \( u \) and \( v \) in \( B \), the vertical distance \( d(u,v) \) in \( B \) is the smallest integer \( k \) such that there is a vertical line segment whose endpoints are at \( u \) and \( v \) passing through \( k \) other bars. Dean et al. [3,4] defined that a graph \( G \) is a \( \text{bar } k \)-visibility graph if \( G \) can be represented as a set of disjoint bars such that any two vertices are adjacent in \( G \) if and only if \( d(u,v) \leq k \), where \( u \) and \( v \) are the associated bars with those vertices. Given a bar \( k \)-visibility graph, we called the corresponding representation a \( \text{bar } k \)-visibility representation. The cases with \( k \) equals 0 and 1 are illustrated in Figure 1. Bar 0-visibility graphs are also known as the bar visibility graphs [2,5]. For \( k = \infty \), bar \( k \)-visibility graphs are exactly the interval graphs (see, for

![Figure 1: A bar 0-visibility graph, a bar 1-visibility graph, and their common representation.](image-url)
We denote $\mathcal{B}_k$ as the family of bar $k$-visibility graphs. Felsner and Massow \cite{6,7} defined that a graph $G$ is a \textit{semi bar $k$-visibility graph} if $G$ can be represented as a set of disjoint bars whose left endpoints have the same $x$-coordinates such that any two vertices are adjacent in $G$ if and only if $d(u, v) \leq k$, where $u$ and $v$ are the associated bars with those vertices. The corresponding representation is called a \textit{semi bar $k$-visibility representation}. The case with $k = 1$ is illustrated in Figure 2. We denote $\mathcal{S}_k$ as the family of semi bar $k$-visibility graphs. The thickness $\theta(G)$ of a graph $G$ is the minimum number of planar subgraphs whose union is $G$ (see, for example, \cite{14}). For any family of graphs $\mathcal{G}$, let $\theta(\mathcal{G}) := \max_{G \in \mathcal{G}} \theta(G)$.

The goal of this paper is to study the thickness of bar $k$-visibility graphs and semi bar $k$-visibility graphs. For the special case when $k = 1$, Dean et al. \cite{3,4} proved that $\theta(\mathcal{B}_1) \leq 4$, and conjectured that $\theta(\mathcal{B}_1) \leq 2$, which was disproved by Felsner and Massow \cite{6,7}. Felsner and Massow also gave a constructive proof for $\theta(\mathcal{S}_1) = 2$. In this paper, we focus on $\theta(\mathcal{B}_k)$ and $\theta(\mathcal{S}_k)$ for general $k$. Dean et al. \cite{3,4} gave the best previously upper bound $3k(6k + 1)$ on $\theta(\mathcal{B}_k)$. We reduce the upper bound to $3k + 3$. It is known that $\theta(\mathcal{B}_k)$ is at least $[(2k + 3)/3]$ as Dean et al. proved that complete graph $K_{4k+4}$ is in $\mathcal{B}_k$. Hence our first result is asymptotically optimal. We also give the first upper bound $2k$ on $\theta(\mathcal{S}_k)$. Felsner and Massow \cite{6,7} proved that complete graph $K_{2k+1}$ is in $\mathcal{S}_k$, so $\theta(\mathcal{S}_k)$ is at least $[(2k + 5)/6]$. Hence our second result is asymptotically optimal.

Table 1 compares previous work and our results. In summary, we prove the following theorem.

\textbf{Theorem 1.}

1. If $G$ is a bar $k$-visibility graph, then $\theta(G) \leq 3k + 3$ for any $k \geq 0$.
2. If $G$ is a semi bar $k$-visibility graph, then $\theta(G) \leq 2k$ for any $k \geq 1$.

\textbf{The importance of the problem.} Mansfield \cite{9} proved that determining the thickness of a graph is NP-hard. The class of graphs whose thickness is known is few—for example, complete graphs and hypercubes (see \cite{10}). If we know better upper bound on the thickness of the graph, then in VLSI design, we can embed the graph using fewer layers \cite{1}. In the scheduling of multihop radio networks, Ramanathan and Lloyd \cite{12,13} gave an approximation algorithm

<table>
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<th>$\theta(\mathcal{B}_k)$</th>
<th>$\theta(\mathcal{S}_k)$</th>
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<tr>
<td>$k = 1$</td>
<td>$\leq 4$</td>
<td>$\geq 2$</td>
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<tr>
<td>$k \geq 1$</td>
<td>$\leq 3k(6k + 1)$</td>
<td>$\geq [(2k + 5)/6]$</td>
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Table 1: Previous work and our result.

![Figure 2: A semi bar 1-visibility graph with its representation.](image)
for the schedule where the number of time slots is bounded by a function of the thickness of a graph.

**Related work on the problem.** Dean et al. [3,4] study bar $k$-visibility graphs and gave bounds on the maximum number of edges and chromatic number of $B_k$. Hartke et al. [8] improved the result by sharpening the bound on maximum number of edges. Hartke et al. also provided some other facts about bar $k$-visibility graphs. They proved that complete graph $K_{4k+4}$ is indeed the largest complete graph in $B_k$, as conjectured by Dean et al. [3,4]; they constructed some forbidden induced subgraphs of the class $B_k$; and they discussed regular bar $k$-visibility graphs. Felsner and Massow [6,7] gave bounds on semi bar $k$-visibility graphs, and gave bounds of chromatic number, clique number, maximum number of edges, and connectivity on $S_k$. They proved that $K_{2k+3}$ is the largest complete graph that can be in $S_k$. Also the yproved that the upper bounds on geometric thickness of $S_1$ is also at most 2. Given a semi bar $k$-visibility graph and an order of bars corresponding to the nodes, Felsner and Massow gave a method to reconstruct a semi bar $k$-visibility representation.

## 2 Bar $k$-visibility graph

Given a graph $G$, $V(G)$ is the node set of $G$ and $E(G)$ is the edge set of $G$. Denote $n_G$ the number of nodes in $G$ and $m_G$ the number of edges in $G$. Consider graph $G$ in $B_k$. If $R$ is a bar $k$-visibility representation of $G$, we denote $G$ as $G(R)$, and the bar in $R$ which corresponds to vertex $x$ in $G$ by $b_x$ or $b(x)$.

### 2.1 Weak bar $k$-visibility graph

A graph $G$ is a weak bar $k$-visibility graph if $G$ is a subgraph of a bar $k$-visibility graph. The case with $k = 1$ is illustrated in Figure 3. We denote $W_k$ as the family of weak bar $k$-visibility graphs.

**Lemma 2.1.** If $G \in W_k$, then there is a graph $H \in B_k$, such that $n_G = n_H$ and $G$ is a subgraph of $H$.

**Proof.** Suppose that $G'$ is a bar $k$-visibility graph and $G$ is a subgraph of $G'$. Let $R'$ be a bar $k$-visibility representation of $G'$, and $R^* = R' - B$, where $B$ is the set of the associated bars of the vertices in $V(G') - V(G)$. Since for every vertex pair $(u,v)$ where $u \in V(G)$ and $v \in V(G')$, if $d(b_u, b_v) \leq k$ in $R'$, then $d(b_u, b_v) \leq k$ in $R^*$, we know that for every edge $e \in E(G)$, $e \in E(G(R^*))$. Hence $G$ is a subgraph of $G(R^*)$ and $n_G = n_{G(R^*)}$. □

**Lemma 2.2** (Hartke et al. [8]). If $G \in B_k$ and $n_G \geq 2k + 2$, then $m_G \leq (k + 1)(3n_G - 4k - 6)$.

![Figure 3: A weak bar 1-visibility graph with its supergraph, and the bar 1-visibility representation of the supergraph.](image-url)
2.2 Arboricity

The arboricity $arb(G)$ of a graph $G$ is the minimum number of forests whose union is $G$, (see, for example, [14]). We know that

$$\theta(G) \leq arb(G) \quad (1)$$

holds for any graph $G$, because the thickness of a forest is one.

Lemma 2.3 (Nash-Williams [11]). For any graph $G$,

$$arb(G) = \max \left\{ \left\lfloor \frac{m_H}{n_H - 1} \right\rfloor : H \subseteq G, n_H > 1 \right\}.$$

2.3 Proof of Theorem 1.1

Proof. Consider any subgraph $H$ of $G$. We have the following two cases.

- Case 1: $1 < n_H < 2k + 2$.
  Since the number of edges for every simple graph with $n$ nodes is at most $\binom{n}{2}$, we have
  $$\frac{m_H}{n_H - 1} \leq \frac{n_H \cdot (n_H - 1)/2}{n_H - 1} = \frac{n_H}{2} < k + 1.$$

- Case 2: $n_H \geq 2k + 2$.
  By the definition of $W_k$ and Lemma 2.1, there exists a graph $H' \in B_k$, such that $n_H = n_{H'}$ and $H$ is a subgraph of $H'$. Hence we know $m_H \leq m_{H'}$. By Lemma 2.2, we know $m_{H'} \leq (k + 1)(3n_{H'} - 4k - 6)$. Therefore,
  $$\frac{m_H}{n_H - 1} \leq \frac{m_{H'}}{n_{H'} - 1} \leq \frac{(k + 1)(3n_{H'} - 4k - 6)}{n_{H'} - 1} \leq \frac{(k + 1)(3n_{H} - 4k - 6)}{n_{H} - 1} = 3(k + 1) - \frac{4k^2 + 7k + 3}{n_{H} - 1} \leq 3k + 3.$$

It follows from Lemma 2.3, that we know $arb(G) \leq 3k + 3$. By (1), we have $\theta(G) \leq 3k + 3$. \qed

3 Semi bar $k$-visibility graph

3.1 Semi bar exactly $k$-visibility graph

A graph $G$ is a semi bar exactly $k$-visibility graph if $G$ can be represented as a set of disjoint bars whose left endpoints have the same $x$-coordinates such that any two vertices are adjacent in $G$ if and only if $d(u, v) = k$, where $u$ and $v$ are the associated bars with those vertices. The case with $k = 1$ is illustrated in Figure 4. We denote $SE_k$ as the family of semi bar exactly $k$-visibility graphs. The outdegree $\deg^+(v)$ of a vertex $v$ is the number of outward directed edges from $v$ (see, for example, [14]).

Lemma 3.1. If $G \in SE_k$, then there is an orientation of edges of $G$ such that for every vertex $v$, $\deg^+(v) \leq 2$. 

Proof. We denote the length of bar $b$ by $\ell(b)$. We label the edges of $G$ by $1, 2, \ldots, m_G$, then we orient the edges of $G$ from 1 to $m_G$ according to the following rule: let $R$ be a semi bar exactly $k$-visibility representation of $G$. For each $j = 1, \ldots, m_G$, let edge $e_j = (x_j, y_j)$. If $\ell(b(x_j)) \geq \ell(b(y_j))$ in $R$, then we give $e_j$ the orientation from $y_j$ to $x_j$, otherwise we give $e_j$ the orientation from $x_j$ to $y_j$. We name the graph $G^*$. For any vertex $v$, suppose that there are more than two bars $b_1, b_2, \ldots, b_q$ which are above $b_v$, such that for each $i$ with $1 \leq i \leq q$, $d(b_i, b_v) = k$ and the orientation of the edges in $G^*$ corresponding to $(b_i, b_v)$ is pointed out from $v$. Let two of those bars be $b_s$ and $b_t$, $\ell(b_s) \geq \ell(b_t)$ and $\ell(b_s) \geq \ell(b_t)$, so every vertical line segment whose endpoints are at $b_s$ and $b_t$ has to pass through $b_i$. Hence $d(b_s, b_v) \neq d(b_t, b_v)$, which is a contradiction. Therefore, there is at most one bar which is above $b_v$, such that the orientation of the edge in $G^*$ corresponding to the bar pair is pointed out from $v$. Similarly, there is at most one bar which is below $b_v$, such that the orientation of the edge in $G^*$ corresponding to the bar pair is pointed out from $v$. So, $\deg^+(v) \leq 2$. \hfill $\Box$

Lemma 3.2. If $G$ admits an orientation such that $\deg^+(v) \leq d$ for every vertex $v$, then $\theta(G) \leq d$.

Proof. By this orientation, we label the outgoing edges of every vertex by $1, 2, \ldots, d$. Let $E_i$ be the set of the edges labeled $i$, and $G_i = (V(G), E_i)$ for each $i$ with $1 \leq i \leq d$, then we know for any component in $G_i$ for each $i$ with $1 \leq i \leq d$, the number of edges in the component is at most the number of nodes in the component, because $G_i$ has an orientation, such that for every vertex $v$, $\deg^+(v) \leq 1$. Hence $\theta(G_i) = 1$ for each $i$ with $1 \leq i \leq d$. Since $\bigcup_{i=1}^d E_i = E(G)$ and $E_i \cap E_j = \emptyset$ for any indices $i$ and $j$ with $i \neq j$, we have

$$\theta(G) \leq \sum_{i=1}^d \theta(G_i) = \sum_{i=1}^d 1 = d. \hfill \Box$$

Lemma 3.3 (Felsner and Massow [7]). If $G \in S_1$, then $\theta(G) \leq 2$.

3.2 Proof of Theorem 1.2

Proof. Suppose that $R$ is a semi bar $k$-visibility representation of $G$. Let

$$E_i = \{(x, y) : d(b_x, b_y) = i\},$$

$$G_i = (V(G), E_i).$$

We have $G_i \in S_{E_i}$ for each $i$ with $0 \leq i \leq k$, and $\bigcup_{i=0}^k E_i = E(G)$. By Lemma 3.1 and Lemma 3.2, we know $\theta(G_i) \leq 2$ for each $i$ with $0 \leq i \leq k$. By the definitions of $S_k$ and $S_{E_k}$, we know $G_0 \cup G_1 \in S_1$. By Lemma 3.3, $\theta(G_0 \cup G_1) \leq 2$. Therefore,

$$\theta(G) \leq \theta(G_0 \cup G_1) + \sum_{i=2}^k \theta(G_i) \leq 2 + 2(k - 1) = 2k. \hfill \Box$$
References


